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Entire Solutions of Higher Order Abstract Cauchy Problems*

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We consider the abstract Cauchy problem for the higher order differential equation

$$\begin{aligned} u^{(n)}(t) + \sum_{i=0}^{n-1} A_i u^{(i)}(t) &= 0, \qquad t \geq 0 \\ u^{(j)}(0) &= u_j, \qquad 0 \leq j \leq n-1, \end{aligned} \tag{$*$}$$

where $A_0, \dots, A_{n-1}$ are closed linear operators in a complex Banach space E. It is shown that if there exist $\theta \in (0, \pi/2)$, $\phi \in (-\pi, \pi]$, $r > 0$ and a positive integer h such that $\lambda^n R_\lambda$ and $\lambda^{-h} A_k R_\lambda$ $(0 \leq k \leq n-1)$ are bounded in $\{\lambda \in \mathbf{C};\ |\arg(e^{-i\phi}\lambda)| \leq \pi/2 + \theta,\ |\lambda| \geq r\}$, where $R_\lambda = (\lambda^n + \sum_{i=0}^{n-1} \lambda^i A_i)^{-1}$, then there exists a dense subset G of E^n such that for every initial value $(u_0, \dots, u_{n-1}) \in G$, $(*)$ has a unique entire solution, that is, a solution which can be extended analytically to the whole complex plane. As a corollary, we get that the Cauchy problems for the parabolic higher order linear differential equations have a unique entire solution for a dense set of initial data. © 1997 Academic Press

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1. INTRODUCTION AND PRELIMINARIES

Let E be a complex Banach space, $n = 2, 3, \ldots$. Consider the following higher order abstract Cauchy problems

$$\begin{aligned} u^{(n)}(t) + \sum_{i=0}^{n-1} A_i u^{(i)}(t) &= 0, \qquad t \geq 0 \\ u^{(j)}(0) &= u_j, \qquad 0 \leq j \leq n-1, \end{aligned} \tag{1.1}$$

where $A_0, \ldots, A_{n-1}$ are closed linear operators with ranges and domains $D(A_0), \ldots, D(A_{n-1})$ contained in E.

In this paper, $R^+ = [0, \infty)$ stands for the nonnegative real number set, **N** the positive integer set, and **C** the complex plane. We denote by $B(E)$ the space of bounded linear operators from E to E. We write

$$D_0 = D(A_0) \cap \cdots \cap D(A_{n-1}),$$

$$P_\lambda = \lambda^n + \sum_{i=0}^{n-1} \lambda^i A_i,$$

$$\rho(A_0, \ldots, A_{n-1}) = \left\{\lambda \in \mathbf{C};\, P_\lambda \text{ is invertible and } P_\lambda^{-1} \in B(E)\right\},$$

$$R_\lambda = P_\lambda^{-1}, \qquad \text{for every } \lambda \in \rho(A_0, \ldots, A_{n-1}),$$

$$\Sigma_\phi(\theta, r) = \left\{z \in \mathbf{C};\, |\arg(e^{-i\phi} z)| \leq \frac{\pi}{2} + \theta, |z| \geq r\right\},$$

$$\left(\theta \in \left(0, \frac{\pi}{2}\right), \phi \in (-\pi, \pi]\right).$$

$[D_0]$ is the Banach space D_0 with the "joint graph norm"

$$\|u\|_0 = \sum_{i=0}^{n-1} \|A_i u\| + \|u\|.$$

In 1942, Hille [7] studied and characterized the analyticity of the strongly continuous semigroup in a Banach space. His result brought to light for the first time the intrinsic relationship between the infinitesimal generator of an analytic semigroup and its resolvent set, estimates of its resolvent. A characterization of the infinitesimal generator of an analytic semigroup in terms of estimates of its resolvent for only real values was obtained by Crandall *et al.* [1]. Kato [8] also gave a different type of characterization of an analytic semigroup based on the behavior of the semigroup near its spectral radius. It is known (see, e.g., [3, 4, 6]) that a strongly continuous semigroup is just the propagator of the following well-posed first-order linear abstract Cauchy problem in Banach space

$$\begin{aligned} u'(t) + Au(t) &= 0, \qquad t \geq 0 \\ u'(0) &= u_0, \end{aligned} \tag{1.2}$$

where $-A$ is the generator of the semigroup. Therefore, the characterizations of the analytic semigroups above are also those of the analytic propagators for the well-posed first-order Cauchy problems.

Concerning the higher order abstract Cauchy problem (1.1), Obrecht [9] obtained a sufficient condition for the analyticity of the propagators of (1.1). Recently, we [12] improved this result and gave a Hille-type characterization of the analytic propagators of the complete second-order abstract Cauchy problem, which is dependent only on the properties of the coefficient operators of (1.1). Further, we obtained this type of result for the case of arbitrary order (see [11] or [13]).

The analyticity of solutions means that they can be extended analytically to a sector in **C**, but not with certainty to the whole complex plane. deLaubenfels [2] discussed first the existence and uniqueness of entire solutions, which can be extended analytically to the entire complex plane **C**, for first-order abstract Cauchy problems and gave some criteria. Following [2], in the present paper, we introduce the concept of the entire solution of (1.1) and investigate its existence and uniqueness. Our purpose is to find some conditions ensuring that Eqs. (1.1) have a unique entire solution for every initial datum in a dense set. These conditions turn out to be particularly satisfied for parabolic higher order linear differential equations (a large and important class of abstract differential equations (see, e.g., [5, 9, 14] and references therein)).

DEFINITION 1.1. A function $u(\cdot) \in C^n(R^+, E)$ is said to be a solution of (1.1) if $A_i u^{(i)}(t) \in C(R^+, E)$ $(0 \leq i \leq n-1)$ and (1.1) is satisfied. We call $u(\cdot)$ an entire solution of (1.1) if $u(\cdot)$ is a solution of (1.1) and it can be extended analytically to the whole complex plane, and $A_i u^{(i)}(\cdot)$ $(0 \leq i \leq n-1)$ are also entire functions.

2. THE RESULTS AND PROOFS

THEOREM 2.1. *Suppose that $A_0, \ldots, A_{n-1}$ are closed linear operators on E with D_0 dense in E, and satisfy the following condition*:

(i) *For some $\theta \in (0, \pi/2)$, $\phi \in (-\pi, \pi]$, $r > 0$,*

$$\rho(A_0, \ldots, A_{n-1}) \supset \Sigma_\phi(\theta, r). \tag{2.1}$$

(ii) *There exist constants* $M > 0$, $h \in \mathbf{N}$ *such that for* $\lambda \in \Sigma_\phi(\theta, r)$,

$$\|\lambda^n R_\lambda\| \leq M, \tag{2.2}$$

$$\|A_k R_\lambda\| \leq M|\lambda|^h, \qquad 0 \leq k \leq n-1. \tag{2.3}$$

Then there exists a dense subset G *of the product space* $E^n = E \times E \times \cdots \times E$ *such that for every initial value* $(u_0, u_1, \ldots, u_{n-1}) \in G$, *the Cauchy problem* (1.1) *has a unique entire solution* $u(\cdot)$: $\mathbf{C} \to [D_0]$.

Proof. Fix $a > r$ and b with $1 < b < (\pi/2)(\pi/2 - \theta)^{-1}$. Let $(a-\lambda)^b$ be the branch of the power function which is holomorphic off the half-line $[a, \infty)$ and positive for $\lambda < a$. For each $z \in \mathbf{C}$, $0 \leq k \leq n-1$, $u \in D_0$, $\varepsilon > 0$, set

$$W_k(z;\varepsilon)u = \frac{e^{i\phi}}{2\pi i}\int_\Gamma \exp\left(ze^{i\phi}\lambda - \varepsilon(a-\lambda)^b\right) \times R_{e^{i\phi}\lambda}\left((e^{i\phi}\lambda)^{n-k-1}u + \sum_{j=k+1}^{n-1}(e^{i\phi}\lambda)^{j-k-1}A_j u\right)d\lambda, \tag{2.4}$$

where Γ is the boundary of $\Sigma_0(\theta, r)$ and is oriented in a way that $\operatorname{Im}\lambda$ increases along Γ (see Fig. 2.1). Clearly, if λ is in the sector

$$\Omega = \left\{z \in \mathbf{C};\, z \neq 0,\, \frac{\pi}{2} + \theta \leq \arg z \leq \frac{3}{2}\pi - \theta\right\},$$

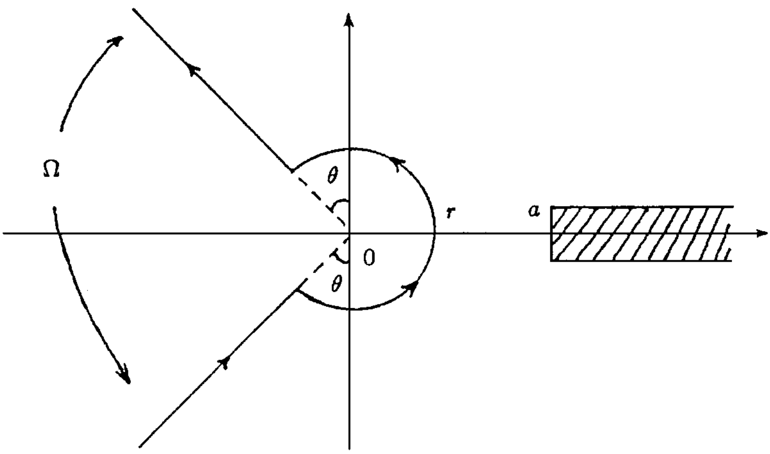

FIG. 2.1.

then $\lambda - a \in \Omega$. Since

$$\operatorname{Re}(a-\lambda)^b = |a-\lambda|^b \cos(b \arg(a-\lambda)) \geq |a-\lambda|^b \cos\left(b\left(\frac{\pi}{2}-\theta\right)\right), \qquad \lambda \in \Omega,$$

we have that for every $\lambda \in \Omega$,

$$|\exp\left(ze^{i\phi}\lambda - \varepsilon(a-\lambda)^b\right)| \tag{2.5}$$

$$\leq \exp\left(|z|\,|\lambda| - \varepsilon \cos\left(b\left(\frac{\pi}{2}-\theta\right)\right)|a-\lambda|^b\right). \tag{2.6}$$

On the other hand, (2.2) implies that for each $u \in D_0$,

$$R_{e^{i\phi}\lambda}\left((e^{i\phi}\lambda)^{n-k-1}u + \sum_{j=k+1}^{n-1}(e^{i\phi}\lambda)^{j-k-1}A_j u\right)$$

is polynomially bounded (for λ). According to this fact, (2.6) and $1 < b < (\pi/2)(\pi/2-\theta)^{-1}$, we obtain that for each $z \in \mathbf{C}$, $0 \leq k \leq n-1$, $u \in D_0$, $\varepsilon > 0$, the integral in (2.4) exists and it defines an entire function of z. Differentiating (2.4) in z up to l times, we get

$$W_k^{(l)}(z;\varepsilon)u = \frac{e^{i\phi}}{2\pi i}\int_\Gamma \exp\left(ze^{i\phi}\lambda - \varepsilon(a-\lambda)^b\right) \times R_{e^{i\phi}\lambda}\left((e^{i\phi}\lambda)^{n+l-k-1}u + \sum_{j=k+1}^{n-1}(e^{i\phi}\lambda)^{j+l-k-1}A_j u\right) d\lambda$$

$$(z \in \mathbf{C}, l \in \mathbf{N}, 0 \leq k \leq n-1, u \in D_0, \varepsilon > 0). \tag{2.7}$$

By (2.3), for each $0 \leq l, k \leq n-1$, $u \in D_0$,

$$A_l R_{e^{i\phi}\lambda}\left((e^{i\phi}\lambda)^{n+l-k-1}u + \sum_{j=k+1}^{n-1}(e^{i\phi}\lambda)^{j+l-k-1}A_j u\right)$$

is also polynomially bounded (for λ in Γ). Hence, it follows from (2.7) and the closedness of A_l that

$$A_l W_k^{(l)}(z;\varepsilon)u = \frac{e^{i\phi}}{2\pi i}\int_\Gamma \exp\left(ze^{i\phi}\lambda - \varepsilon(a-\lambda)^b\right) \times A_l R_{e^{i\phi}\lambda}\left((e^{i\phi}\lambda)^{n+l-k-1}u + \sum_{j=k+1}^{n-1}(e^{i\phi}\lambda)^{j+l-k-1}A_j u\right) d\lambda$$

$$(z \in \mathbf{C}, 0 \leq l, k \leq n-1, u \in D_0, \varepsilon > 0). \tag{2.8}$$

Thus,

$$
\begin{aligned}
W_k^{(n)}&(z;\varepsilon)u + \sum_{l=0}^{n-1} A_l W_k^{(l)}(z;\varepsilon)u \\
&= \frac{e^{i\phi}}{2\pi i} \lim_{T\to\infty} \int_{\Gamma_T} \exp\left(ze^{i\phi}\lambda - \varepsilon(a-\lambda)^b\right) \\
&\qquad \times \left((e^{i\phi}\lambda)^{n-k-1} + \sum_{j=k+1}^{n-1} (e^{i\phi}\lambda)^{j-k-1} A_j u \right) d\lambda, \qquad (2.9)
\end{aligned}
$$

where $\Gamma_T = \Gamma \bigcap \{z \in \mathbf{C}; |z| \le T\}$. Since the integral in (2.9) is analytic in Ω, we can shift the path of the integral to the arc

$$\left\{ Te^{i\alpha};\ \frac{\pi}{2} + \theta \le \alpha \le \frac{3}{2}\pi - \theta \right\}$$

using the well-known Cauchy theorem. Thus, combining (2.6), we have

$$
\begin{aligned}
&\left\| \int_{\Gamma_T} \exp\left(ze^{i\phi}\lambda - \varepsilon(a-\lambda)^b\right) \right. \\
&\quad \left. \times \left((e^{i\phi}\lambda)^{n-k-1} + \sum_{j=k+1}^{n-1} (e^{i\phi}\lambda)^{j-k-1} A_j u \right) d\lambda \right\| \\
&\qquad \le \int_{\frac{\pi}{2}+\theta}^{\frac{3}{2}\pi-\theta} \exp\left(|z|\,|T| - \varepsilon \cos\left(b\left(\frac{\pi}{2} - \theta \right) \right) (T-a)^b \right) \\
&\qquad\quad \times \left(\|u\|\,|T|^{n-k} + \sum_{j=k+1}^{n-1} \|A_j u\|\,|T|^{j-k} \right) d\alpha \\
&\qquad \to 0, \qquad \text{as } T \to \infty. \qquad (2.10)
\end{aligned}
$$

Consequently, for any $\varepsilon > 0$, $v_k \in D_0$ $(0 \le k \le n-1)$,

$$u_\varepsilon(t) = \sum_{k=0}^{n-1} W_k(t;\varepsilon) v_k \qquad (2.11)$$

is a solution of (1.1) with initial value

$$u_\varepsilon^{(l)}(0) = \sum_{k=0}^{n-1} C_{\varepsilon,k}^l v_k,$$

where

$$
\begin{aligned}
C_{\varepsilon,k}^{l} u = \frac{e^{i\phi}}{2\pi i} \int_{\Gamma} & \exp\left(-\varepsilon(a-\lambda)^{b}\right)\left(e^{i\phi}\lambda\right)^{l-k-1} \\
& \times \left[u - \sum_{j=0}^{k} \left(e^{i\phi}\lambda\right)^{j} R_{e^{i\phi}\lambda} A_j u \right] d\lambda, \\
& \qquad \varepsilon > 0, 0 \le k, l \le n-1, u \in D_0. \quad (2.12)
\end{aligned}
$$

Now, observe that a deformation of contour as in the treatment of (2.10) shows

$$
\begin{aligned}
& \frac{e^{i\phi}}{2\pi i} \int_{\Gamma} \exp\left(-\varepsilon(a-\lambda)^{b}\right)\left(e^{i\phi}\lambda\right)^{l-k-1} v_k \, d\lambda \\
& \quad = \begin{cases} 0, & \text{if } l-k-1 \ge 0, \\ \dfrac{\left(e^{i\phi}\right)^{l-k}}{(k-l)!} \left[\dfrac{d^{k-l}}{d\lambda^{k-l}} \exp\left(-\varepsilon(a-\lambda)^{b}\right) v_k \right]_{\lambda=0}, & \text{if } l-k-1 < 0. \end{cases}
\end{aligned}
$$

On the other hand, shifting the path Γ of integral to the arc

$$
\left\{ Te^{i\alpha}; -\frac{\pi}{2} - \theta \le \alpha \le \frac{\pi}{2} + \theta \right\}
$$

instead and according to (2.2), we see that for $v \in E$, $j \in \mathbf{N}$ and $j \le n-2$,

$$
\begin{aligned}
& \left\| \frac{e^{i\phi}}{2\pi i} \lim_{\varepsilon \to 0} \int_{\Gamma} \exp\left(-\varepsilon(a-\lambda)^{b}\right)\left(e^{i\phi}\lambda\right)^{j} R_{e^{i\phi}\lambda} v \, d\lambda \right\| \\
& \quad \le \frac{1}{2\pi} \lim_{T \to \infty} \int_{-\frac{\pi}{2}-\theta}^{\frac{\pi}{2}+\theta} MT^{-1} \|v\| \, d\alpha = 0.
\end{aligned}
$$

Hence, we have

$$
\lim_{\varepsilon \to 0} C_{\varepsilon,k}^{l} v_k = \begin{cases} 0, & \text{if } k \ne l, \\ v_l, & \text{if } k = l, \end{cases}
$$

and therefore

$$
\lim_{\varepsilon \to 0} u_{\varepsilon}^{(l)}(0) = v_l, \qquad 0 \le l \le n-1.
$$

This ends the proof of existence, because of the denseness of D_0.

Now, we show the uniqueness.

Let $u(\cdot)$ be an entire solution of (1.1) with the initial values $u_j = 0$, $0 \le j \le n-1$. Clearly, there exists $\alpha \in (-\pi/2, \pi/2)$ such that $\lambda e^{i\alpha} \in \Sigma_\phi(\theta, r)$ for each $\lambda > r$. If we define

$$M_t = \sup\{\|u(e^{-i\alpha}s)\|; t-1 \le s \le t\}, \qquad t > 1,$$

then we have

$$\begin{aligned}\left\| \int_{t-1}^{t} \exp(\lambda(t-s-1)) u(e^{-i\alpha}s)\, ds \right\| &\le M_t \int_{t-1}^{t} \exp(\lambda(t-s-1))\, ds \\ &= M_t \lambda^{-1}(1-e^{-\lambda}) \to 0, \qquad \text{as } \lambda \to \infty. \end{aligned} \tag{2.13}$$

On the other hand, integrating by parts, we get that for each $t \ge 0$, $\lambda > r$, $1 \le k \le n$,

$$\begin{aligned}&(e^{i\alpha}\lambda)^k \int_0^t \exp(\lambda(t-s)) u(e^{-i\alpha}s)\, ds \\ &\quad = -\sum_{l=0}^{k-1} e^{i(k-l)\alpha}\lambda^{k-l-1} u^{(l)}(e^{-i\alpha}t) + \int_0^t \exp(\lambda(t-s)) u^{(k)}(e^{-i\alpha}s)\, ds.\end{aligned}$$

Thus, it follows from $u^{(n)}(e^{-i\alpha}s) + \sum_{k=0}^{n-1} A_k u^{(k)}(e^{-i\alpha}s) = 0$ $(s \ge 0)$ that for $\lambda > r$,

$$\begin{aligned}&\left\| \int_0^{t-1} \exp(\lambda(t-s-1)) u(e^{-i\alpha}s)\, ds \right\| \\ &\quad = \left\| e^{-\lambda} R_{e^{i\alpha}\lambda} P_{e^{i\alpha}\lambda} \int_0^{t-1} \exp(\lambda(t-s)) u(e^{-i\alpha}s)\, ds \right\| \\ &\quad = \Bigg\| -e^{-\lambda} \Bigg\{ \sum_{l=0}^{n-1} e^{i(k-l)\alpha}\lambda^{n-l-1} R_{e^{i\alpha}\lambda} u^{(n)}(e^{-i\alpha}t) \\ &\qquad\qquad + \sum_{k=0}^{n-1}\sum_{l=0}^{k-1} e^{i(k-l)\alpha}\lambda^{k-l-1} R_{e^{i\alpha}\lambda} A_k u^{(l)}(e^{-i\alpha}t) \\ &\qquad\qquad + \int_{t-1}^{t} \exp(\lambda(t-s)) u(e^{-i\alpha}s)\, ds \Bigg\} \Bigg\|. \end{aligned} \tag{2.14}$$

By (2.13), (2.14), and (2.2), we obtain that for every given $t \ge 1$,

$$\lim_{\lambda\to\infty} \int_0^{t-1} \exp(\lambda s) u\big(e^{-i\alpha}(t-s-1)\big)\, ds = 0.$$

In view of [10, Lemma 4.1.1], we have $u(e^{-i\alpha}(t-s-1)) = 0$ $(0 \le s \le t-1)$. Since t is arbitrary, $u(e^{-i\alpha}t) = 0$ for any $t \ge 0$. Therefore $u(z) = 0$, $z \in \mathbf{C}$. This ends the proof of the theorem.

Remark 2.2. Equation (2.11) together with (2.4) and (2.12) provides an explicit expression of the entire solution of (1.1).

Recall that the parabolicity of the equation in (1.1) means that for some $\theta \in (0, \pi/2)$, $r > 0$,

$$\rho(A_0, \ldots, A_{n-1}) \supset \Sigma_0(\theta, r)$$

and there is $M > 0$ such that

$$\|\lambda^n R_\lambda\|, \|\lambda^k A_k R_\lambda\| \le M \qquad (0 \le k \le n-1).$$

As a direct consequence, we have

COROLLARY 2.3. *Assume that the differential equation in* (1.1) *is parabolic. Then the conclusion in Theorem* 2.1 *holds.*

Remark 2.4. Conditions ensuring that the second-order equation (resp. higher order equation) in a Banach space be parabolic were studied in [5] (resp. [14]). Corollary 2.3 enables us to apply the results in [5, 14] to obtain entire solutions for corresponding abstract Cauchy problems.

3. EXAMPLES

EXAMPLE 3.1. Let q, k, l, $h \in \mathbf{N}$ with $k < 2q$, $l < 4q$, $h < 6q$, let $b > 1$, and let $a_i(\cdot)$ $(i = 0, 1, 2)$ be a bounded measurable function defined in m-dimensional Euclidean space R^m. By Δ, we denote the Laplacian $\sum_{i=1}^m \partial^2/\partial x_i^2$. Then the Cauchy problem

$$\frac{\partial^3 u}{\partial t^3} + \left((-1)^q \Delta^q + a_2(x) \sum_{i=1}^m \frac{\partial^k}{\partial x_i^k} \right) \frac{\partial^2 u}{\partial t^2} + \left(b\Delta^{2q} + a_1(x) \sum_{i=1}^m \frac{\partial^l}{\partial x_i^l} \right) \frac{\partial u}{\partial t}$$
$$+ \left((-1)^q \Delta^{3q} + a_0(x) \sum_{i=1}^m \frac{\partial^h}{\partial x_i^h} \right) u = 0, \qquad (t, x) \in R^+ \times R^m,$$

$$u(0,x) = u_0(x), \frac{\partial u}{\partial t}(0,x) = u_1(x), \frac{\partial^2 u}{\partial t^2}(0,x) = u_2(x), x \in R^m,$$

has a unique solution $u \in C^\infty(R^+, H^{6q}(R^m))$, which can be extended to an entire function: $\mathbf{C} \to H^{6q}(R^m)$, for every initial value (u_0, u_1, u_2) in a dense subset of $L^2(R^m) \times L^2(R^m) \times L^2(R^m)$.

Proof. Take $E = L^2(R^m)$. Let

$$B_0 = (-1)^q \Delta^{3q} \qquad \text{with } D(B_0) = H^{6q}(R^m),$$

$$B_1 = b\Delta^{2q} \qquad \text{with } D(B_1) = H^{4q}(R^m),$$

$$B_2 = (-1)^q \Delta^{q} \qquad \text{with } D(B_2) = H^{2q}(R^m),$$

$$S_0 = a_0(x) \sum_{i=1}^{m} \frac{\partial^h}{\partial x_i^h} \qquad \text{with } D(S_0) = H^h(R^m),$$

$$S_1 = a_1(x) \sum_{i=1}^{m} \frac{\partial^l}{\partial x_i^l} \qquad \text{with } D(S_1) = H^l(R^m),$$

$$S_2 = a_2(x) \sum_{i=1}^{m} \frac{\partial^k}{\partial x_i^k} \qquad \text{with } D(S_2) = H^k(R^m).$$

Clearly, B_0 is a densely defined and strictly nonnegative operator on E (see [14, Definition 1.3]), and

$$B_1 = bB_0^{2/3}, \qquad B_2 = B_0^{1/3}.$$

Since $k < 2q$, $l < 4q$, $h < 6q$, we have by virtue of the *Moment inequality* that there exists a constant $C > 0$ such that

$$\|S_0 u\| \leq C\|u\|^{1-h/6q}\|B_0 u\|^{h/6q}, \qquad \text{for } u \in D(B_0),$$

$$\|S_1 u\| \leq C\|u\|^{1-1/4q}\|B_1 u\|^{1/4q}, \qquad \text{for } u \in D(B_1),$$

$$\|S_2 u\| \leq C\|u\|^{1-k/2q}\|B_2 u\|^{k/2q}, \qquad \text{for } u \in D(B_2).$$

Define $A_i = B_i + S_i$, $i = 0, 1, 2$. Then, Theorem 2.1 combined with Theorems 2.6 and 3.2 in [14] shows the result desired.

EXAMPLE 3.2. Let $E = L^2(0,1)$, $a(x) \in C^1[0,1]$ with $a(x) \neq 0$ for each $x \in [0,1]$, $\int_0^1 a^{-1}(\xi)\, d\xi \neq 0$, and

$$a(x) \in S_\phi(\theta) = \left\{z \in \mathbf{C}; |\arg(e^{-i\phi}z)| \leq \theta\right\}, \qquad \text{for any } x \in [0,1],$$

for some $\theta \in (0, \pi/2)$, $\phi \in (-\pi, \pi]$. Let

$$A_0 = -\frac{\partial^2}{\partial x^2}\cdot, \qquad A_1 = -\frac{\partial}{\partial x}\left(a(x)\frac{\partial}{\partial x}\cdot\right)$$

with

$$D(A_0) = D(A_1) = \left\{u \in H^2(0,1); u(x)\mid_{x=0,1} = 0\right\}.$$

We have that $0 \in \rho(A_1)$. In fact, for any $v(\cdot) \in E$,

$$u(x) = \left[\left(\int_0^1 a^{-1}(\xi)\,d\xi \right)^{-1} \int_0^1 a^{-1}(\xi) \int_0^\xi v(\eta)\,d\eta\,d\xi \right] \int_0^x a^{-1}(\xi)\,d\xi$$
$$- \int_0^x a^{-1}(\xi) \int_0^\xi v(\eta)\,d\eta\,d\xi \tag{3.1}$$

is in $D(A_1)$ and it is the solution of $A_1 u = v$, and it is clear from (3.1) that

$$\|A_1^{-1} v\| \le \text{const}\|v\|.$$

On the other hand, for each $u \in D(A_1)$,

$$\langle A_1 u, u \rangle = - \int_0^1 \frac{\partial}{\partial x} \left(a(x) \frac{\partial}{\partial x} u(x) \right) \bar{u}(x)\,dx$$
$$= \int_0^1 a(x) |u'(x)|^2\,dx \in S_\phi(\theta).$$

We thus obtain by virtue of [10, Theorem 3.9, p. 12] that

$$\rho(-A_1) \supset \bigcup_{0<\beta<\frac{\pi}{2}} \Sigma_\phi\left(\frac{\pi}{2} - \beta, 0 \right)$$

and there exists $C_0 > 0$ such that

$$\|(\lambda + A_1)^{-1}\| \le C_0 |\lambda|^{-1}, \qquad \lambda \in \Sigma_\phi\left(\frac{\pi}{4} - \frac{\theta}{2}, 1 \right). \tag{3.2}$$

Taking $\lambda_0 \in \rho(-A_1)$, then $A_0(\lambda_0 + A_1)^{-1} \in L(E)$ since $D(A_0) = D(A_1)$. It follows from (3.2) that there exists $r > 1$ such that for $\lambda \in \Sigma_\phi(\pi/4 - \theta/2, r)$,

$$\|\lambda^{-1} A_0(\lambda + A_1)^{-1}\| \le \|A_0(\lambda_0 + A_1)^{-1}\|\, \|\lambda^{-1}(\lambda_0 + A_1)(\lambda + A_1)^{-1}\| < \tfrac{1}{2}.$$

Thus, from the equality

$$\left(\lambda^2 + \lambda A_1 + A_0\right)^{-1} = \lambda^{-1}(\lambda + A_1)^{-1}\left[I + \lambda^{-1} A_0(\lambda + A_1)^{-1} \right]^{-1},$$
$$\lambda \in \Sigma_\phi\left(\frac{\pi}{4} - \frac{\theta}{2}, r \right),$$

we get that for λ as above, $\lambda \in \rho(A_0, A_1)$ and

$$\|\lambda^2\left(\lambda^2 + \lambda A_1 + A_0\right)^{-1}\|, \|\lambda A_1\left(\lambda^2 + \lambda A_1 + A_0\right)^{-1}\| \le C$$

for some constant $C > 0$. Now applying Theorem 2.1, we claim that the Cauchy problem for the damped wave equation

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial^2}{\partial t \partial x}u(t,x)\right) - \dfrac{\partial^2}{\partial x^2}u(t,x) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t > 0, 0 < x < 1), \\ u(t,x)\mid_{z=0,1} = 0 \qquad\qquad\qquad\qquad\qquad (t \geq 0) \\ u(0,x) = u_0(x), \dfrac{\partial u}{\partial t}(0,x) = u_1(x) \qquad (0 \leq x \leq 1) \end{cases}$$

has a unique solution $u \in C^\infty(R^+, H^2(0,1))$, which can be extended to an entire function: $\mathbf{C} \to H^2(0,1)$, for every initial value (u_0, u_1) in a dense subset of $L^2(0,1) \times L^2(0,1)$.

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